Optimal control problems on stratifiable state constraints sets.

Cristopher Hermosilla, Hasnaa Zidani

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ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Introduction

We consider an infinite horizon problem with state constraints $\mathcal{K}$:

$$(P) \quad \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \;\middle|\; \begin{array}{c} u : [0, +\infty) \to \mathcal{A} \text{ measurable} \\ y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0 \end{array} \right\}.$$

where $\lambda > 0$ is fixed and $y_{x,u}(\cdot)$ is a trajectory of the control system

$$\begin{cases} \dot{y} = f(y, u) & \text{a.e. } t \geq 0 \\ y(0) = x \in \mathcal{K} \end{cases}$$

We are mainly concerned with a characterization of the value function of (P) as the bilateral solution to a Hamilton-Jacobi-Bellman equation.

Standing Hypothesis (SH)

- $\mathcal{K}$ is closed and $\mathcal{A} \subseteq \mathbb{R}^m$ is nonempty and compact.
- $\ell : \mathbb{R}^N \times \mathcal{A} \to [0, +\infty)$ and $f : \mathbb{R}^N \times \mathcal{A} \to \mathbb{R}^N$ are continuous and Lipschitz w.r.t. the state :

$$\exists L > 0 \text{ such that } \quad \left.\begin{array}{r} |f(x,u) - f(y,u)| \\ |\ell(x,u) - \ell(y,u)| \end{array}\right\} \leq L|x-y| \qquad \forall u \in \mathcal{A}.$$

- We assume convexity of the augmented dynamics :

$$\left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda t}\ell(x,u) + r \end{pmatrix} \;\middle|\; \begin{array}{c} u \in \mathcal{A} \\ r \geq 0 \end{array} \right\}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

The value function

Basic properties

$$v(x) := \inf_{u \in \mathbb{A}(x)} \int_0^{\infty} e^{-\lambda t} \ell(y_{x,u}(t), u(t))dt, \qquad \forall x \in \mathcal{K},$$

where

$$\mathbb{A}(x) = \{u : [0, +\infty) \to \mathcal{A} \text{ measurable } \mid y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0\}.$$

Proposition

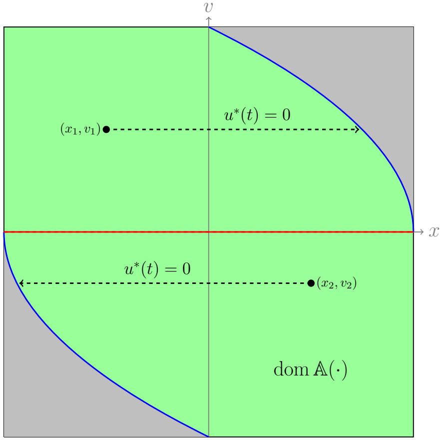
Suppose that (SH) holds. Then, $\exists \lambda_0 = \lambda_0(f, \ell) > 0$ so that if $\lambda > \lambda_0$

- *if $v(x) \in \mathbb{R}$ then there exists $u \in \mathbb{A}(x)$ an optimal control.*
- *$v : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.*
- *v has linear growth on its domain :*

$$\exists c_v > 0 \qquad |v(x)| \leq c_v(1 + |x|) \qquad \forall x \in \operatorname{dom} \mathbb{A}.$$

An example

$$(\mathcal{P}) \qquad \min \int_0^\infty e^{-\lambda t} u(t)^2 dt,$$
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$
$$y_1(0) = x,$$
$$y_2(0) = v,$$
$$u \in \mathcal{A} := [-1, 1]$$
$$y_1(t), y_2(t) \in [-r, r]$$



Dynamic Programming Principle

Proposition

The value function *satisfies the DPP* *: For any* $T > 0$

$$\upsilon(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_{x,u}(t), u(t))dt + e^{-\lambda T} \upsilon(y_{x,u}(T)) \right\}.$$

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Definition

Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be l.s.c.

i) φ *is weakly decreasing provided* $\forall x \in \operatorname{dom} \varphi$, $\exists u \in \mathbb{A}(x)$ *such that*

$$e^{-\lambda t}\varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s))ds \leq \varphi(x) \quad \forall t \geq 0.$$

ii) φ *is strongly increasing provided* $\operatorname{dom} \mathbb{A} \subseteq \operatorname{dom} \varphi$ *and* $\forall x \in \mathcal{K}$, $\forall u \in \mathbb{A}(x)$

$$e^{-\lambda t}\varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s))ds \geq \varphi(x) \quad \forall t \geq 0.$$

The value function

Comparison principle

Proposition

Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be l.s.c. with linear growth.

- *If φ is weakly decreasing, then $\upsilon(x) \leq \varphi(x)$ for all $x \in \mathcal{K}$.*
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Corollary

Suppose that (SH) holds with $\lambda > \lambda_0$. Then $\upsilon(\cdot)$ is the unique l.s.c. function with linear growth defined on $\mathcal{K}$ which is weakly decreasing and strongly increasing at the same time.

Definition (Subdifferentials)

Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. A vector $\zeta \in \mathbb{R}^N$ is called a viscosity subgradient of φ at $x \in \operatorname{dom} \varphi$ if and only if :

$$\exists g \in \mathcal{C}^1(\mathbb{R}^N) \text{ s.t. } \nabla g(x) = \zeta \text{ and } \varphi - g \text{ attains a local minimum at } x.$$

Furthermore, ζ is called a proximal subgradient of φ at x if for some $\sigma > 0$,

$$g(y) := \langle \zeta, y - x \rangle - \sigma |y - x|^2.$$

The set of all proximal subgradients at x is denoted by $\partial_P \varphi(x)$.

Weakly Decreasing Principle

Characterization of supersolutions

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Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Then φ is weakly decreasing if and only if

$$\lambda\varphi(x) + H(x, \zeta) \geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. If φ is strongly increasing then

$$(1) \qquad \lambda\varphi(x) + H(x,\zeta) \leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

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Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. If* φ *is strongly increasing then*

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Remark

- The converse does not hold without additional hypothesis.
- (1) provides information only for trajectories that touch the boundary a finite number of times.
- There is no information of what happens on the boundary.

Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When does (1) become sufficient ? ?

- $v(\cdot)$ is continuous up to the boundary.
- Interior approximation of trajectories.
- Some monotonicity properties of the solutions to (1).

Feasible Neighboring Trajectories Approach

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When does (1) become sufficient ? ?

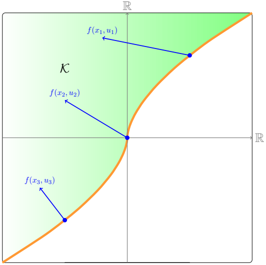
- $v(\cdot)$ is continuous up to the boundary.
- Interior approximation of trajectories.
- Some monotonicity properties of the solutions to (1).

What do we need to achieve one of these ? ?

$\looparrowright$ $\mathcal{K} = \overline{\mathrm{int}(\mathcal{K})}$ and tameness properties.

$\looparrowright$ Pointing Conditions

(Inward or Outward).



Our example

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

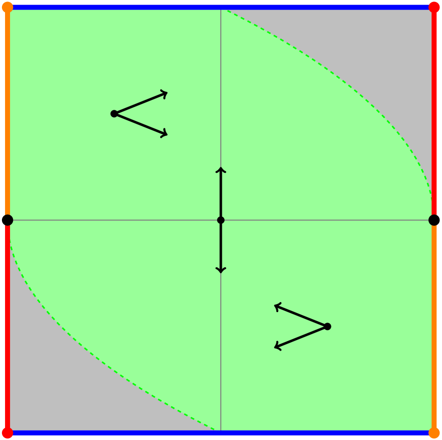
$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$

Note that :

$$\left\langle \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

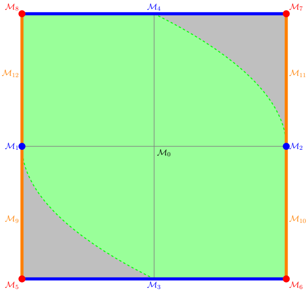


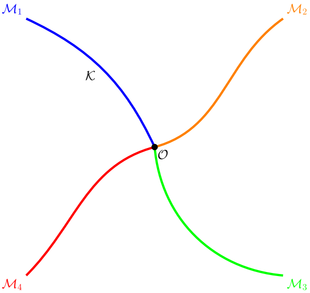
Stratifiable sets

Definition

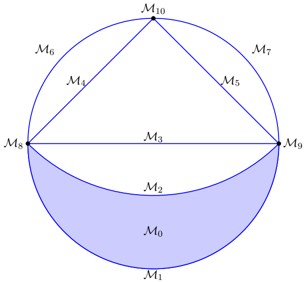
A closed set $\mathcal{K} \subseteq \mathbb{R}^N$ is said to be stratifiable *if there exists a locally finite collection $\{\mathcal{M}_i : i \in \mathcal{I}\}$ of embedded manifolds of $\mathbb{R}^N$ such that :*

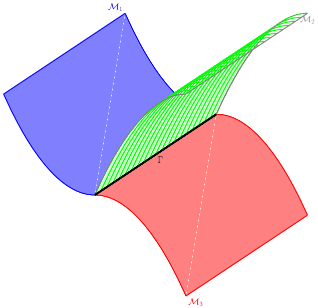
- $\mathcal{K} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$ *and* $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ *whenever* $i \neq j$.
- $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, *then* $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$ *and* $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.





Stratifiable sets





The class of stratifiable sets on $\mathbb{R}^N$ is wide, it includes :

- Semilinear sets $\rightarrow$ finite union of open polyhedra.
- Semialgebraic sets $\rightarrow$ finite union of polynomial manifolds.
- Subanalytic sets $\rightarrow$ locally finite union of analytic manifolds.

Characterization of subsolutions

Some basic definitions and notation

Assume that $\mathcal{K}$ is stratifiable and let $\{\mathcal{M}_i\}$ be its strata. Then, for each stratum $\mathcal{M}_i$ we define

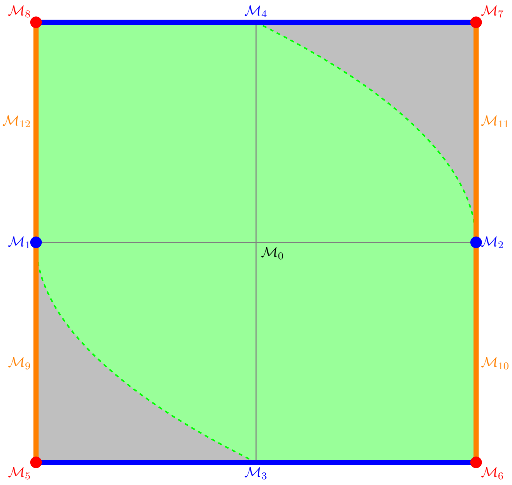
- the set of tangent controls as the map $\mathcal{A}_i : \mathcal{M}_i \rightrightarrows \mathcal{A}$ given by

$$\mathcal{A}_i(x) := \{u \in \mathcal{A} \mid f(x,u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}.$$

- the tangential Hamiltonian as the map $H_i : \mathcal{M}_i \times \mathbb{R}^N \to \mathbb{R}$ given by

$$H_i(x,\zeta) = \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\}.$$

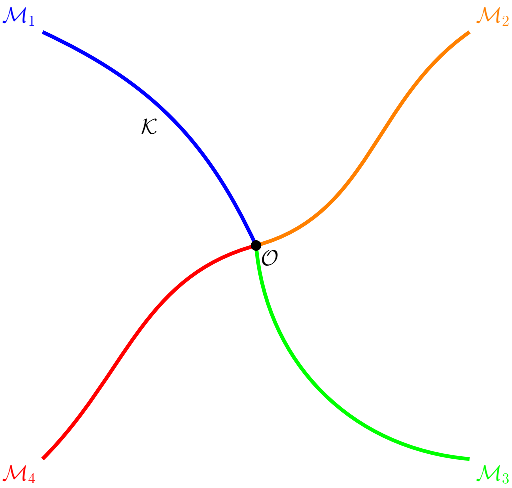
Tangent controls



$$f(x_1, x_2, u) = (x_2, u)$$
$$u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$
$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \ldots, 4$$
$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \ldots, 12$$

Tangent controls



$\exists A_i \subseteq \mathcal{A}$ such that

$$f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i).$$

$$\mathcal{A}_i(x) = \mathcal{A}_i, \quad \forall i = 1, \ldots, 4$$

Let $M_0 = \{\mathcal{O}\}$:

$$\mathcal{A}_0(\mathcal{O}) = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

Characterization of subsolutions

Proposition (CH - Zidani)

Suppose that (SH) holds with $\mathcal{K}$ is stratifiable and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Assume that

(H_0) $\quad \mathcal{A}_i$ *is a Lipschitz set-valued map or has empty images on* $\mathcal{M}_i$.

If φ is strongly increasing then for each $i \in \mathcal{I}$

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x,\zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P \varphi_i(x),$$

where $\varphi_i(x) = \varphi(x)$ if $x \in \overline{\mathcal{M}}_i$ and $+\infty$ otherwise.

Remark

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

The converse ?

Strong Invariance Principle on $\mathrm{int}(\mathcal{K})$

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}} \{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall\zeta \in \partial_P\varphi(x).$$

$$\Downarrow$$

$y_{x,u}(s) \in \mathrm{int}(\mathcal{K})$ for every $s \in (a,b)$, where $0 \leq a < b < +\infty$ then

$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a}\int_a^b e^{-\lambda s}\ell(y_{x,u},u)ds.$$

$$\Downarrow$$

Any admissible trajectory $y_{x,u}$ defined on $[0,T]$ with $y_{x,u}(s) \in \mathrm{int}(\mathcal{K})$ for every $s \in (0,T)$ satisfies the Strong Increasing Inequality.

The converse ?

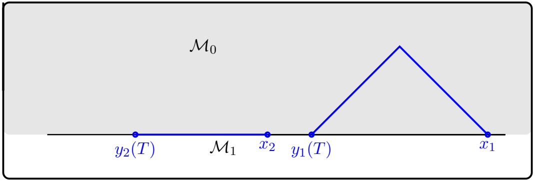
Strong Invariance Principle on each stratum

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}_i(x)} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P\varphi_i(x).$$

$$\Downarrow$$

$y_{x,u}(s) \in \mathcal{M}_i$ for every $s \in (a,b)$, where $0 \leq a < b < +\infty$ then

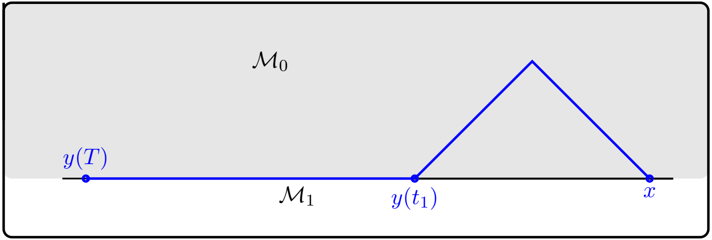
$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a}\int_a^b e^{-\lambda s}\ell(y_{x,u},u)ds.$$



Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \ldots, n\}$ we have

$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda s}\ell(y_{x,u}, u)ds.$$

Therefore $y_{x,u}$ satisfies the Strong Increasing Inequality ! !

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

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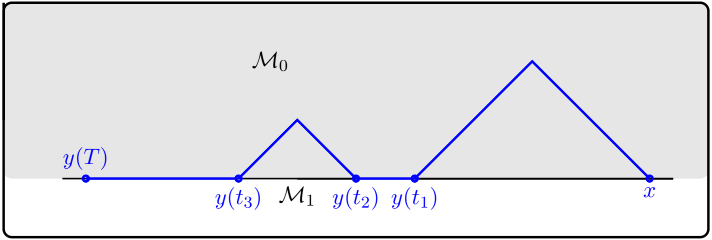

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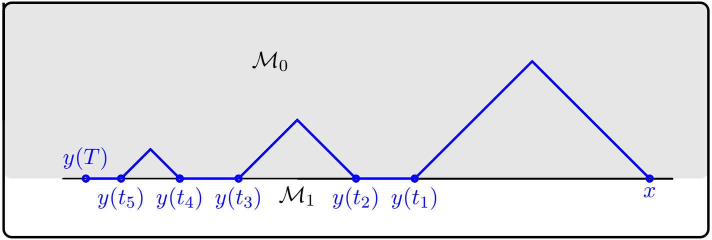

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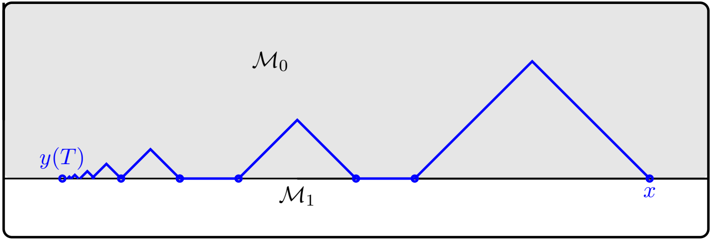
Therefore $y_{x,u}$ satisfies the Strong Increasing Inequality ! !

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there is NO partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.

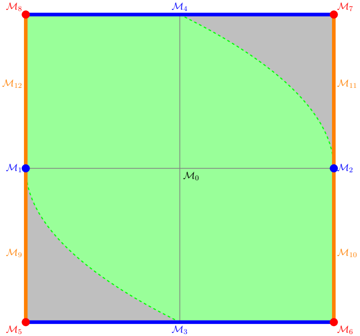


What if the trajectory "chatters" between two or more strata ? ? ?

Controllability assumption

$$(H_1)\quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \operatorname{dom} \mathcal{A}_i \neq \emptyset,\ \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i,\ \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.
$\mathcal{R}_i^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A}_i(x))$.



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Lemma

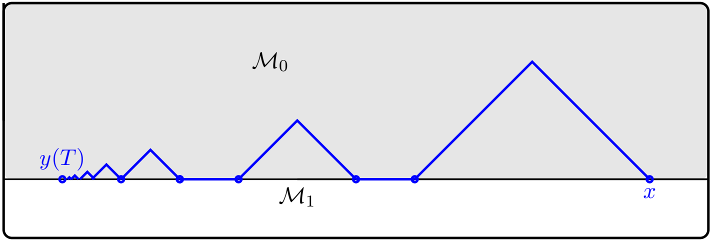
Suppose (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold.
For any $x \in \mathcal{K}$ and $T > 0$ there exists $L > 0$ such that : $\forall u \in \mathbb{A}(x), \forall \varepsilon > 0$ if $y_{x,u}(b), y_{x,u}(a) \in \mathcal{M}_i$ with $0 \leq a < b \leq T$ for some $i \in \mathcal{I}$ then,

$$\varphi(y_{x,u}(a)) \leq e^{\lambda \varepsilon} \left(e^{-\lambda(b-a)} \varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds \right) + L\varepsilon.$$

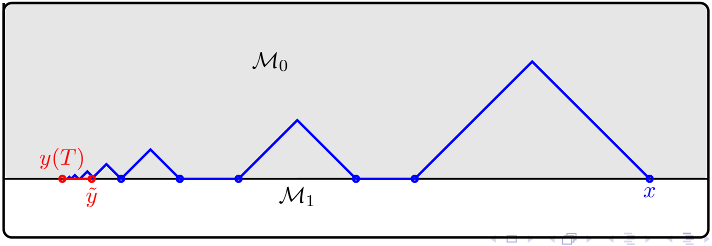
provided $\quad \operatorname{meas}(\{t \in [a, b] \mid y_{x,u}(t) \notin \mathcal{M}_i\}) < \varepsilon.$

Controllability assumption :

Chattering trajectory :



Approximated trajectory :



Characterization of the strong increasing principle

Proposition (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold. Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function with $\mathrm{dom}\,\mathbb{A} \subseteq \mathrm{dom}\,\varphi$ that satisfies

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x,\zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P \varphi_i(x),\ \forall i \in \mathcal{I}.$$

Then φ is strongly increasing.

Recall

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

Characterization of the value function

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\begin{aligned} \lambda v(x) + H(x,\zeta) \geq 0 \qquad & \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H_i(x,\zeta) \leq 0 \qquad & \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P v_i(x),\ \forall i \in \mathcal{I}, \end{aligned}$$

where $v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}$.

Characterization of the value function : $\mathcal{M}_0 = \mathrm{int}(\mathcal{K})$.

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable with $\mathrm{int}(\mathcal{K}) \neq \emptyset$ and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the value function

$$\upsilon(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

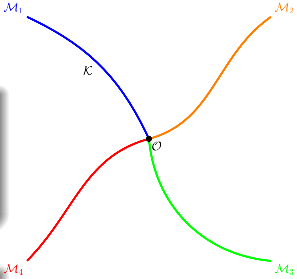
$$\begin{aligned}
\lambda\upsilon(x) + H(x,\zeta) &\geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P \upsilon(x),\\
\lambda\upsilon(x) + H(x,\zeta) &\leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall \zeta \in \partial_P \upsilon(x),\\
\lambda\upsilon(x) + H_i(x,\zeta) &\leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P \upsilon_i(x),\ \forall i \in \mathcal{I} \setminus \{0\},
\end{aligned}$$

where $\upsilon_i(x) = \begin{cases} \upsilon(x) & \text{if } x \in \overline{\mathcal{M}_i} \\ +\infty & \text{otherwise.} \end{cases}$

Applications to Networks

Suppose $\mathcal{K}$ is a network with one junction $\mathcal{O}$ and let $\mathcal{M}_1, \ldots, \mathcal{M}_p$ be its branches.
Assume that for each $i \in \{1, \ldots, p\}$, $\exists \mathcal{A}_i \subseteq \mathcal{A}$ s.t.

$$(H_3) \qquad f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i), \quad \forall x \in \mathcal{M}_i.$$



For instance, for some $v_i \in \mathbb{R}^N \setminus \{0\}$

$$\mathcal{M}_i = (0, +\infty)v_i \text{ and } f(x, u) = f_i(x, u)v_i, \quad \forall x \in \mathcal{M}_i$$

with and f_i real-valued.

Claim

The hypothesis (H_0) and (H_1) are satisfied.

Applications to Networks

Theorem (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ a network as before and $\lambda > \lambda_0$. Assume that (H_3) holds and let

$$\mathcal{A}_0 = \{u \in \mathcal{A} \mid \ f(\mathcal{O}, u) = 0\}.$$

Then the value function is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\lambda v(x) + \max_{u \in \mathcal{A}_i} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\} = 0 \quad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P v(x),$$

$$\lambda v(\mathcal{O}) + H(\mathcal{O}, \zeta) \geq 0 \quad \forall \zeta \in \partial_P v(\mathcal{O}),$$

$$\lambda v(\mathcal{O}) - \min_{u \in \mathcal{A}_0} \ell(\mathcal{O}, u) \leq 0.$$

Final remarks

- The interior of $\mathcal{K}$ can always be taken as a stratum and so, the constrained Hamiton-Jacobi equation proposed by Soner is included in the set of equations proposed in the main theorem.
- The characterize of the value function neither requires its continuity nor that the state constraint set has empty interior.
- Under the continuity of the value function (on its domain) the controllability assumption can be dropped.
- The characterization for networks can be extended to a suitable notion of generalized network where the junction is replaced by a manifold.

Thanks for your attention !